

ANNUAIRE DE L'UNIVERSITÉ DE SOFIA "KLIMENT OHRIDSKI"

FACULTÉ DE MATHÉMATIQUES ET MÉCANIQUE

Tome 78

Livre 1 – Mathématiques

1984

CURVATURE PROPERTIES AND ISOTROPIC PLANES OF RIEMANNIAN AND ALMOST HERMITIAN MANIFOLDS OF INDEFINITE METRICS

Adrijan Borisov, Georgi Ganchev, Ognian Kassabov

The behaviour of the curvature tensor on isotropic planes on a Riemannian manifold of indefinite metric is closely related to the curvature properties of the manifold. The importance of isotropic planes in questions concerning curvature properties of a Riemannian manifold M with indefinite metric firstly has been shown by Dajczer and Nomizu in [1]. Namely, they have shown that M is of constant sectional curvature if and only if the curvature tensor vanishes on all the isotropic planes. In this paper we study two types of isotropic planes: weakly isotropic and strongly isotropic planes. We prove that a Riemannian manifold of indefinite metric is conformally flat if and only if its curvature tensor vanishes on all the strongly isotropic planes. We specialize the plane axiom for Riemannian manifolds of indefinite metrics. We show that manifolds satisfying plane axiom of weakly (strongly) isotropic planes are of constant sectional curvature (conformally flat). Further we study analogous problems on almost Hermitian manifolds of indefinite metrics taking into account both structures: the metric and the almost complex structure.

1. RIEMANNIAN MANIFOLDS OF INDEFINITE METRICS

Preliminaries. Let M be a Riemannian manifold with indefinite metric g of signature $(s, n-s)$, i.e. the tangent space $T_p M$ in $p \in M$ is isometric to \mathbf{R}_s^n with the inner product:

$$\langle x, y \rangle = - \sum_{i=1}^s x^i y^i + \sum_{j=s+1}^n x^j y^j .$$

The curvature operator on M is given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} ,$$

for arbitrary vector fields X, Y on M . The Ricci tensor and the scalar curvature of M are denoted by ρ and τ respectively. A 2-plane (2-dimensional subspace) α of $T_p M$ is said to

be nondegenerate, weakly isotropic or strongly isotropic if the restriction of g on α is of rank 2, 1 or 0 respectively. The sectional curvature of a nondegenerate 2-plane α in $T_p M$ with a basis $\{x, y\}$ is given by

$$K(\alpha) = K(x, y) = \frac{R(x, y, y, x)}{g(x, x)g(y, y) - g^2(x, y)} .$$

A pair $\{x, a\}$ of tangent vectors at a point $p \in M$ is said to be orthonormal of signature $(+, -)$ if $g(x, x) = 1$, $g(a, a) = -1$, $g(x, a) = 0$. Orthonormal pairs of signature $(+, +)$ or $(-, -)$ are determined analogously. Orthonormal quadruples of certain signature are determined in a similar way.

Firstly, the particular importance of degenerate planes in pseudo-Riemannian geometry has been shown by the theorem:

Theorem A [1]. If $R(x, y, y, x) = 0$, whenever $\text{span}\{x, y\}$ is weakly isotropic, then all the nondegenerate 2-planes have the same sectional curvature, i.e. the manifold is of constant sectional curvature.

In fact, this theorem is formulated in [1] for arbitrary degenerate 2-planes, but the proof is also valid for the proposition in the above given form.

A tensor T of type (0,4) over $T_p M$, $p \in M$, is said to be curvature-like tensor if it has the following properties:

- a) $T(x, y, z, u) = -T(y, x, z, u)$,
- b) $T(x, y, z, u) = -T(x, y, u, z)$,
- c) $T(x, y, z, u) + T(y, z, x, u) + T(z, x, y, u) = 0$

for all x, y, z, u in $T_p M$.

We shall use Theorem 1, a) from [1] in the following form:

Theorem B. Let T be a curvature-like tensor over $T_p M$, $p \in M$. If $T(x, y, z, x) = 0$, whenever $\{x, y\}$ is an orthonormal pair of signature $(+, -)$ and $g(x, z) = g(y, z) = 0$, then all the nondegenerate 2-planes have the same sectional curvature with respect to T .

Conformally flat Riemannian manifolds and isotropic planes. The conformal curvature tensor C on a Riemannian manifold is given by

$$C = R - \frac{1}{n-2}\varphi + \frac{\tau}{(n-1)(n-2)}\pi_1 ,$$

where

$$\varphi(x, y, z, u) = g(y, z)\rho(x, u) - g(x, z)\rho(y, u) + g(x, u)\rho(y, z) - g(y, u)\rho(x, z) ,$$

$$\pi_1(x, y, z, u) = g(y, z)g(x, u) - g(x, z)g(y, u) .$$

It is well known, that, if $n > 3$, then M is conformally flat if and only if C vanishes.

The main result in this section is the following theorem, which clarifies the relation between the conformal curvature tensor and the behaviour of R on strongly isotropic 2-planes.

Theorem 1. Let $s \geq 2$, $n - s \geq 2$. Then M is conformally flat if and only if $R(\xi, \eta, \eta, \xi) = 0$, whenever $\text{span}\{\xi, \eta\}$ is strongly isotropic.

Proof. Let $\{x, y, a, b\}$ be an arbitrary orthonormal quadruple of signature $(+, +, -, -)$. Then

$$R(x + a, y + b, y + b, x + a) = 0$$

implies

$$R(x, y, b, a) + R(x, b, y, a) = 0 .$$

Hence, using the first Bianchi identity, we find

$$(1) \quad R(x, y, b, a) = 0 .$$

Replacing $\{x, b\}$ by $\{(x + tb)/\sqrt{1 - t^2}, (tx + b)/\sqrt{1 - t^2}\}$, $|t| < 1$, in (1), we obtain

$$R(x, y, a, x) + R(b, y, a, b) = 0 .$$

From here it is not difficult to derive

$$(2) \quad (n - 2)R(x, y, a, x) - \rho(y, a) = 0 .$$

Replacing y by $(y + tb)/\sqrt{1 - t^2}$, $|t| < 1$, in (2), we find

$$(3) \quad (n - 2)R(x, b, a, x) - \rho(b, a) = 0 .$$

From (2), (3) and Theorem B it follows $C = 0$.

The inverse is a simple verification.

Using Theorem 1 and the continuity of R , in a similar way we have

Theorem 2. Let M ($s \geq 2$, $n - s \geq 2$) be a Riemannian manifold of indefinite metric. Then the following conditions are equivalent:

- 1) M is conformally flat;
- 2) $R(x, y, a, b) = 0$, whenever $\{x, y, a, b\}$ is an orthonormal quadruple of signature $(+, +, -, -)$;
- 3) $K(x, y) + K(a, b) = K(x, a) + K(y, b)$, whenever $\{x, y, a, b\}$ is an orthonormal quadruple of signature $(+, +, -, -)$.

This proposition specifies the characterizations of conformally flat manifolds given by Schouten [2, p.307] and Kulkarni [3].

Plane axioms in the Riemannian geometry of indefinite metric. Let N ($\dim N = m \geq 3$) be a differentiable manifold with a linear connection of zero torsion. In general, N is said to satisfy the axiom of r -planes ($2 \leq r < m$), if, for each point p and for any r -dimensional subspace E of $T_p M$, there exists a totally geodesic submanifold N' containing p such that $T_p N' = E$.

Now we specialize the general plane axiom to the considered manifolds.

A Riemannian manifold M of indefinite metric is said to satisfy the axiom of weakly (strongly) isotropic 2-planes, if, for each point p and for any weakly (strongly) isotropic 2-plane E in $T_p M$, there exists a totally geodesic submanifold M' containing p such that $T_p M' = E$ and all the tangent spaces of M' are weakly (strongly) isotropic in M .

Theorem 3. Let M ($n \geq 3$) be a Riemannian manifold of indefinite metric. If M satisfies the axiom of weakly isotropic 2-planes, then M is of constant sectional curvature.

Proof. Let ξ be an isotropic vector in $T_p M$ and $x \perp \xi$, such that $g(x, x) = 1$ or $g(x, x) = -1$. By the conditions of the theorem, there exists a totally geodesic submanifold M' of M through p such that $T_p M' = \text{span}\{\xi, x\}$. Since M' is totally geodesic and the

connection is symmetric, $R(\xi, x)x$ is tangent to M' at p . This implies $R(\xi, x, x, \xi) = 0$. According to Theorem A, M is of constant sectional curvature.

Theorem 4. Let M ($s \geq 2, n - s \geq 2$) be a Riemannian manifold of indefinite metric. If M satisfies the axiom of strongly isotropic 2-planes, then M is conformally flat.

Proof. Let ξ, η be arbitrary isotropic perpendicular vectors in $T_p M$, $p \in M$. By the condition of the theorem, there exists a totally geodesic submanifold M' of M containing p such that $T_p M' = \text{span}\{\xi, \eta\}$. Then $R(\xi, \eta)\eta$ is in $T_p M'$. Hence, $R(\xi, \eta, \eta, \eta, \xi) = 0$. Applying Theorem 1, we obtain M is conformally flat.

2. ALMOST HERMITIAN MANIFOLDS OF INDEFINITE METRICS

Preliminaries. In this part M will stand for an almost Hermitian manifold with indefinite metric g of signature $(2s, 2(n - s))$ and almost complex structure J , i.e.

$$J^2 = -id, \quad g(JX, JY) = g(X, Y)$$

for arbitrary vector fields X, Y on M .

If $\nabla J = 0$, then M is a Kaehlerian manifold of indefinite metric.

A 2-plane α in $T_p M$ is said to be holomorphic (antiholomorphic) if $J\alpha = \alpha$ ($J\alpha \perp \alpha$, $J\alpha \neq \alpha$). A pair $\{x, y\}$ of vectors in $T_p M$ is said to be holomorphic (antiholomorphic) if $\text{span}\{x, y\}$ is a holomorphic (antiholomorphic) 2-plane. Antiholomorphic triples and quadruples are determined in a similar way.

The manifold M is said to be of pointwise constant holomorphic (antiholomorphic) sectional curvature if in each point $p \in M$ all the nondegenerate holomorphic (antiholomorphic) 2-planes have the same sectional curvature, which is a function of the point. As in the definite case, if M is a Kaehler manifold of pointwise constant holomorphic (antiholomorphic) sectional curvature, then M is of constant holomorphic sectional curvature.

Let $\{e_i, i = 1, \dots, 2n\}$, be an orthonormal basis of $T_p M$. The Ricci $*$ -tensor and the scalar $*$ -curvature are given by

$$\rho^*(y, z) = \sum_{i=1}^{2n} g(e_i, e_i) R(e_i, y, Jz, J e_i),$$

$$\tau^* = \sum_{i=1}^{2n} g(e_i, e_i) \rho^*(e_i, e_i)$$

respectively.

The proof of the next proposition, given in [4] is also valid in the indefinite case.

Theorem C. Let T be a curvature-like tensor of type (0,4) over $T_p M$. If

- 1) $T(x, Jx, Jx, x) = 0$, for an arbitrary vector x in $T_p M$,
- 2) $T(x, y, y, x) = 0$, whenever $\text{span}\{x, y\}$ is an antiholomorphic 2-plane,
- 3) $T(x, Jx, y, x) = 0$, whenever $\text{span}\{x, y\}$ is an antiholomorphic 2-plane,

then $T = 0$.

Almost Hermitian manifolds of pointwise constant antiholomorphic sectional curvature. In this section we give an analogue of Theorem A for almost Hermitian manifolds of indefinite metric.

Theorem 5. Let M be an almost Hermitian manifold of indefinite metric and $n \geq 3$. If $R(X, \xi, \xi, X) = 0$, whenever $\text{span}\{X, \xi\}$ is a weakly isotropic antiholomorphic 2-plane, then M is of pointwise constant antiholomorphic sectional curvature.

Proof. Let $n - s \geq 2$ (the case $s \geq 2$ is treated analogously). We choose x, y, a in $T_p M$, $p \in M$ so that $\{x, y, a\}$ is an orthonormal antiholomorphic triple of signature $(+, +, -)$. From the condition of the theorem we have

$$R(x, y + a, y + a, x) = 0 ,$$

From here we find

$$(4) \quad K(x, y) = K(x, a) ,$$

$$(5) \quad R(x, y, a, x) = 0 .$$

Replacing a by $(a + Ja)/\sqrt{2}$ in (4) we get

$$(6) \quad R(x, a, Ja, x) = 0 .$$

Analogously

$$(7) \quad R(x, y, Jy, x) = 0 .$$

Now, let Y, Z be arbitrary unit vectors in $T_p M$ so that $Y, Z \perp x, Jx$. If Y, Z have the same signature, then (4) implies

$$(8) \quad K(x, Y) = K(x, Z)$$

in the following way. If $\{Y, Z\}$ has a signature $(+, +)$, we choose $a \perp x, Jx, Y, JY, Z, JZ$ and then $K(x, Y) = K(x, a) = K(x, Z)$ according to (4). If $\{Y, Z\}$ has a signature $(-, -)$, we choose $y \perp x, Jx, Y, JY, Z, JZ$ and further $K(x, Y) = K(x, y) = K(x, Z)$. If Y, Z are of different signature, then (4) - (7) imply (8). Similarly, if $g(a, a) = -1$, we find

$$(9) \quad K(a, Y) = K(a, Z)$$

for unit vectors Y, Z in $T_p M$ and $Y, Z \perp a, Ja$.

Finally, let α, β be arbitrary nondegenerate antiholomorphic 2-planes with orthonormal bases $\{X, Y\}$ and $\{Z, U\}$, respectively, and let $E = \text{span}\{X, JX\}$. If $\{Z, U\}$ is of signature $(+, -)$, we choose a vector $W \in T_p M, W \perp Z, JZ, g(W, W) = 1$. Then $\{Z, W\}$ is of signature $(+, +)$ and

$$K(\beta) = K(Z, U) = K(Z, W)$$

taking into account (8). Hence we can assume $\{Z, U\}$ is of signature $(+, +)$ or $(-, -)$. Let Z' be a unit vector and $Z' \in \beta \cap E^\perp$. Choosing $U' \in \beta, U' \perp Z'$ and using (8), (9), we obtain

$$K(\alpha) = K(X, Y) = K(X, Z') = K(Z', U') = K(\beta) .$$

This gives: M is of pointwise constant antiholomorphic sectional curvature.

Remark. If M ($n \geq 3$) is an almost Hermitian manifold of pointwise constant antiholomorphic sectional curvature ν , then its curvature tensor has the form [5]

$$R - \frac{1}{2(n+1)}\psi(\rho^*) + \frac{\tau^*}{(2n+1)(2n+2)}\pi_2 = \nu \left(\pi_1 - \frac{1}{2n+1}\pi_2 \right),$$

where

$$\begin{aligned} \psi(S)(x, y, z, u) = & g(y, Jz)S(x, Ju) - g(x, Jz)S(y, Ju) - 2g(x, Jy)S(z, Ju) \\ & + g(x, Ju)S(y, Jz) - g(y, Ju)S(x, Jz) - 2g(z, Ju)S(x, Jy) \end{aligned}$$

is a curvature like-tensor whenever $S(x, Jy) + S(y, Jx) = 0$ and

$$\pi_2(x, y, z, u) = g(y, Jz)g(x, Ju) - g(x, Jz)g(y, Ju) - 2g(x, Jy)g(z, Ju).$$

Then, the direct verification shows the inverse proposition of Theorem 5.

If M is Kaehlerian and $n \geq 3$, the following conditions are equivalent [8]

- 1) M is of constant holomorphic sectional curvature μ ;
- 2) M is of constant antiholomorphic sectional curvature $\mu/4$.

Then Theorem 5 implies

Corollary [6]. Let M be a Kaehler manifold of indefinite metric and $n \geq 3$. If $R(X, \xi, \xi, X) = 0$, whenever $\text{span}\{X, \xi\}$ is a weakly isotropic antiholomorphic 2-plane, then M is of constant holomorphic sectional curvature.

Almost Hermitian manifolds with vanishing Bochner curvature tensor. Firstly, we shall prove

Lemma 1. Let T be a curvature-like tensor of type (0,4) over $T_p M$. If

- 1) $T(x, Jx, Jx, x) = 0$, whenever $g(x, x) = 1$,
 - 2) $T(x, a, a, x) = 0$, whenever $\text{span}\{x, a\}$ is an antiholomorphic 2-plane of signature $(+, -)$,
 - 3) $T(x, Jx, b, x) = 0$, whenever $\text{span}\{x, b\}$ is an antiholomorphic 2-plane of signature $(+, -)$,
- then $T = 0$.

Proof. We shall show that the conditions 1), 2) and 3) imply the corresponding conditions of Theorem C. For instance, let 1) hold good. If $\alpha \in T_p M$ and $g(a, a) = -1$, we choose $x \perp a$, Ja and $g(x, x) = 1$. Then 1) implies

$$T(x + ta, Jx + tJa, Jx + tJa, x + ta) = 0$$

for every $|t| < 1$. From here we find $T(a, Ja, Ja, a) = 0$ and hence $T(X, JX, JX, X) = 0$ for arbitrary non-isotropic vector X . Further, approximating any isotropic vector ξ with non-isotropic vectors, we obtain $T(\xi, J\xi, J\xi, \xi) = 0$ and the condition 1) of Theorem C. The conditions 2) and 3) in Theorem C follow in a similar way.

The Bochner curvature tensor $B(R)$ for an almost Hermitian manifold M ($2n \geq 6$) is given by the equality [7]

$$\begin{aligned} B(R) = & R - \{16(n+2)\}^{-1}(\varphi + \psi)(\rho + 3\rho^*)(R + \overline{R}) \\ & - \{16(n-2)\}^{-1}(3\varphi - \psi)(\rho - \rho^*)(R + \overline{R}) - \{(4(n+1))^{-1}\psi(\rho^*) - (4(n-1))^{-1}\varphi(\rho)\}(R - \overline{R}) \\ & + \{16(n+1)(n+2)\}^{-1}(\tau + 3\tau^*)(R)(\pi_1 + \pi_2) \end{aligned}$$

$$+\{16(n-1)(n-2)\}^{-1}(\tau - \tau^*)(R)(3\pi_1 - \pi_2) ,$$

where $\overline{R}(x, y, z, u) = R(Jx, Jy, Jz, Ju)$ for all $x, y, z, u \in T_p M$.

The next theorem gives a characterization of almost Hermitian manifolds of indefinite metric with vanishing Bochner curvature tensor.

Theorem 6. Let M ($s \geq 2$, $n-s \geq 2$) be an almost Hermitian manifold of indefinite metric. The following conditions are equivalent:

- 1) $R(\xi, \eta, \eta, \xi) = 0$, whenever $\text{span}\{\xi, \eta\}$ is a strongly isotropic antiholomorphic 2-plane,
- 2) $B(R) = 0$.

Proof. Let ξ be an arbitrary isotropic vector in $T_p M$. Choosing an isotropic vector η , so that $\eta \perp \xi$, $J\xi$, from the condition we find

$$R(\xi, J\xi + \eta, J\xi + \eta, \xi) = 0 .$$

From here we obtain

$$R(\xi, J\xi, J\xi, \xi) = 0 .$$

This equality gives

$$R(x + a, Jx + Ja, Jx + Ja, x + a) = 0$$

for an arbitrary orthonormal pair $\{x, a\}$ of signature $(+, -)$. Using the last equality we get

$$(10) \quad K(x, Jx) + K(a, Ja) = K(x, Ja) + K(Jx, a) - 2R(x, Jx, Ja, a) - 2R(x, Ja, Jx, a).$$

Let y be in $T_p M$, so that $\{x, y, a\}$ is an orthonormal antiholomorphic triple of signature $(+, +, -)$. Replacing a by $(a + ty)/\sqrt{1 - t^2}$, $|t| < 1$ in (10) we find

$$(11) \quad K(x, Jx) + K(y, Jy) = K(x, Jy) + K(Jx, y) + 2R(x, Jx, Jy, y) + 2R(x, Jy, Jx, y).$$

If $\{e_1, \dots, e_{2n}\}$ is an orthonormal basis of $T_p M$, from (10) and (11) we obtain

$$(12) \quad \sum_{i=1}^{2n} K(e_i, Je_i) = \frac{\tau + 3\tau^*}{2(n+1)} ,$$

$$(13) \quad K(x, Jx) = \frac{1}{2(n+2)} \{\rho(x, x) + \rho(Jx, Jx) + 6\rho^*(x, x)\} - \frac{\tau + 3\tau^*}{4(n+1)(n+2)} .$$

Now, let $\{x, y, a, b\}$ be an orthonormal antiholomorphic quadruple of signature $(+, +, -, -)$. We have

$$R(x + a, y + b, y + b, x + a) = 0$$

and further

$$R(x, y + b, y + b, x) + R(a, y + b, y + b, a) = 0 .$$

From here, for an arbitrary $z \in T_p M$, $z \perp y, Jy, a, Ja, b, Jb$ we have

$$R(x, y + b, y + b, x) - R(z, y + b, y + b, z) = 0 .$$

The last two equalities imply

$$(14) \quad \begin{aligned} 2(n-2)R(x, y + b, y + b, x) &= \rho(y + b, y + b) - R(Jy, y + b, y + b, Jy) \\ &\quad + R(Jb, y + b, y + b, Jb) . \end{aligned}$$

Firstly, from (14) it follows

$$2(n-2)\{K(x, y) - K(x, b)\} = \rho(y, y) + \rho(b, b) - K(y, Jy) + K(b, Jb) + K(Jy, b) - K(y, Jb) .$$

Using (12), from this equality we obtain

$$\begin{aligned} & (4n^2 - 14n + 11)K(x, b) + (2n - 3)K(x, Jb) + K(Jx, Jb) - K(Jx, b) \\ & + 2(n - 1)\{K(x, Jx) + K(b, Jb)\} = (2n - 3)\rho(x, x) - 2(n - 2)\rho(b, b) \\ & + \rho(Jx, Jx) - 2\rho(Jb, Jb) - \tau + \frac{\tau + 3\tau^*}{2(n + 1)} . \end{aligned}$$

This formula gives

$$\begin{aligned} & 4(n - 1)(n - 2)K(x, b) + 2(n - 1)\{K(x, Jx) + K(b, Jb)\} \\ & = (2n - 3)\{\rho(x, x) - \rho(b, b)\} + \rho(Jx, Jx) + \rho(Jb, Jb) - \tau + \frac{\tau + 3\tau^*}{2(n + 1)} . \end{aligned}$$

From here, taking into account (13), we find

$$\begin{aligned} (15) \quad K(x, b) &= \frac{2n^2 - 5}{4(n - 1)(n^2 - 4)}\{\rho(x, x) - \rho(b, b)\} + \frac{3}{4(n - 1)(n^2 - 4)}\{\rho(Jx, Jx) \\ & - \rho(Jb, Jb)\} - \frac{3}{2(n^2 - 4)}\{\rho(x, x) - \rho(b, b)\} - \frac{2n^2 + 3n + 4}{8(n^2 - 1)(n^2 - 4)}\tau \\ & + \frac{9n}{8(n^2 - 1)(n^2 - 4)}\tau^* . \end{aligned}$$

Further, (14) implies

$$2(n - 2)R(x, y, b, x) = \rho(y, b) - R(Jy, y, b, Jy) + R(Jb, y, b, Jb) .$$

Replacing here x by $(Jx - Jy)/\sqrt{2}$ and y by $(x + y)/\sqrt{2}$ we get

$$\begin{aligned} & 2(n - 2)R(y, Jy, Jy, b) - (2n - 5)\{R(x, Jx, Jy, b) + R(Jy, x, b, Jx)\} \\ & - R(b, Jb, Jb, y) + R(Jx, y, b, Jx) = \rho(y, b) , \end{aligned}$$

from where it follows

$$(16) \quad \begin{aligned} & 2nR(y, Jy, Jy, b) - R(b, Jb, Jb, y) - 3R(b, Jb, y, b) \\ & = \rho(y, b) + 3\rho^*(b, y) . \end{aligned}$$

for an arbitrary orthonormal pair $\{y, b\}$ of signature $(+, -)$.

Changing in (15) x by $(y + tb)/\sqrt{1 - t^2}$ and b by $(tJy + Jb)/\sqrt{1 - t^2}$ for arbitrary $|t| < 1$ we check

$$(17) \quad 2(n - 1)\{R(Jy, y, y, Jb) + R(Jb, y, b, Jb)\} = -\rho(y, b) + \rho(Jy, Jb) .$$

Finally, from

$$R(y + b, Jy + Jb, Jy + Jb, y + b) = 0$$

we derive

$$(18) \quad R(y, Jy, Jy, b) + R(b, Jb, Jb, y) + R(y, Jy, Jb, y) + R(b, Jb, Jy, b) = 0 .$$

From (16), (17) and (18) it follows easily

$$(19) \quad \begin{aligned} R(y, Jy, Jy, b) = & -\frac{3}{4(n-1)(n+2)}\rho(y, b) + \frac{2n+1}{4(n-1)(n+2)}\rho(Jy, Jb) \\ & -\frac{3}{4(n+1)(n+2)}\rho^*(b, y) + \frac{3(2n+3)}{4(n+1)(n+2)}\rho^*(y, b) . \end{aligned}$$

Taking into account (13), (15) and (19) and Lemma 1 we obtain $B(R) = 0$.

The inverse is a straightforward verification.

Lemma 2. Let M ($s \geq 2$, $n - s \geq 2$) be a Kaehlerian manifold of indefinite metric. The following conditions are equivalent:

- 1) $R(\xi, J\xi, J\xi, \xi) = 0$, whenever ξ is an isotropic vector,
- 2) $R(\xi, \eta, \eta, \eta, \xi) = 0$, whenever $\text{span}\{\xi, \eta\}$ is a strongly isotropic antiholomorphic 2-plane.

Proof. The implication 2) \rightarrow 1) was shown in the proof of Theorem 6 in the case of an arbitrary almost Hermitian manifold.

Now, let M satisfy 1). This condition gives

$$R(\xi + \eta, J\xi + J\eta, J\xi + J\eta, \xi + \eta) = 0 ,$$

which implies

$$2R(\xi, J\xi, J\eta, \eta) + 2R(\xi, J\eta, J\xi, \eta) + R(\xi, J\eta, J\eta, \xi) + R(\eta, J\xi, J\xi, \eta) = 0 .$$

Using this equality and the first Bianchi identity we find

$$R(\xi, \eta, \eta, \xi) + 3R(\xi, J\eta, J\eta, \xi) = 0 .$$

Now 2) follows in a straightforward way.

Applying Lemma 2 and Theorem 6 we obtain

Theorem 7. Let M ($s \geq 2$, $n - s \geq 2$) be a Kaehlerian manifold of indefinite metric. The following conditions are equivalent:

- 1) $R(\xi, J\xi, J\xi, \xi) = 0$, whenever ξ is an isotropic vector,
- 2) $B(R) = 0$.

Remark. In [8] there is announced the following proposition without a proof:

Theorem 6.5. Let M be a connected indefinite Kaehler manifold with complex dimension $n \geq 2$ and index $2s > 0$. If $R(u, Ju, Ju, u) = 0$ for all isotropic vectors $u \in T_p M$ holds, then M has constant holomorphic sectional curvature.

Plane axioms for almost Hermitian manifolds of indefinite metrics. Let M be an almost Hermitian manifold of indefinite metric. M is said to satisfy the axiom of the weakly (strongly) isotropic antiholomorphic 2-planes, if, for each point p in M and for any weakly (strongly) isotropic antiholomorphic 2-plane E in $T_p M$, there exists a totally geodesic submanifold M' of M containing p , such that $T_p M' = E$ and all the tangent spaces of M' are weakly (strongly) isotropic antiholomorphic 2-planes in M . The axiom of isotropic holomorphic 2-planes is formulated in a similar way. We note, that every isotropic holomorphic 2-plane is necessarily strongly isotropic.

In this section we discuss these axioms. We need the following propositions.

Theorem D. Let M be an almost Hermitian manifold of indefinite metric. If

$$R(x, Jx, Jx, a) = 0 ,$$

whenever $\{x, a\}$ is an orthonormal antiholomorphic pair of signature $(+, -)$, then M is of pointwise constant holomorphic sectional curvature and $R = \overline{R}$.

This proposition can be derived in a similar way as the corresponding theorem in the case of an almost Hermitian manifold of definite metric [9].

Using Lemma 1, we obtain immediately.

Lemma 3. Let M ($s \geq 2, n - s \geq 2$) be an almost Hermitian manifold of indefinite metric. If M is of pointwise constant holomorphic sectional curvature μ , pointwise constant antiholomorphic sectional curvature ν , and $R = \overline{R}$, then the curvature tensor has the form:

$$(20) \quad R = \nu\pi_1 + \frac{\mu - \nu}{3}\pi_2 .$$

Theorem E. [7] Let M be an almost Hermitian manifold and $2n \geq 6$. If the curvature tensor of M has the form (20), then M is of constant sectional curvature or M is a Kaehler manifold of constant holomorphic sectional curvature.

The proof of this theorem is also valid in the case of an indefinite metric.

Theorem 8. Let M ($s \geq 2, n - s \geq 2$) be an almost Hermitian manifold of indefinite metric. If M satisfies the axiom of weakly isotropic antiholomorphic 2-planes, then M is of constant sectional curvature or M is a Kaehler manifold of constant holomorphic sectional curvature.

Proof. Let $\{\xi, x\}$ be a basis of an weakly isotropic antiholomorphic 2-plane in $T_p M$ with $g(\xi, \xi) = g(\xi, x) = g(\xi, Jx) = 0$ and $g(x, x) = \pm 1$. By the conditions of the theorem, there exists a totally geodesic submanifold M' through p such that $T_p M' = \text{span}\{\xi, x\}$. This implies $R(\xi, x, x, \xi) = 0$. Applying Theorem 5, we obtain M is of pointwise constant antiholomorphic sectional curvature. On the other hand, since $Jx \perp \xi, x$, then $R(\xi, x, x, Jx) = 0$. From here it follows immediately $R(Jx, x, x, a) = 0$, whenever $\{x, a\}$ is an orthonormal antiholomorphic pair of signature $(+, -)$. Applying Theorem D, Lemma 3 and Theorem E we obtain the proposition.

Theorem 9. Let M ($s \geq 2, n - s \geq 2$) be an almost Hermitian manifold of indefinite metric. If M satisfies the axiom of strongly isotropic antiholomorphic 2-planes, then M has vanishing Bochner curvature tensor.

Proof. Let $\{\xi, \eta\}$ be a basis of a strongly isotropic antiholomorphic 2-plane in $T_p M$. From the condition of the assertion we find $R(\xi, \eta)\eta$ is in $\text{span}\{\xi, \eta\}$. Hence, $R(\xi, \eta, \eta, \xi) = 0$. Applying Theorem 6 we obtain $B(R) = 0$.

Theorem 10. Let M ($s \geq 2, n - s \geq 2$) be a Kaehlerian manifold of indefinite metric. If M satisfies the axiom of isotropic holomorphic 2-planes, then M has vanishing Bochner curvature tensor.

Proof. Let ξ be an arbitrary isotropic vector in $T_p M$. By the conditions of the theorem we have $R(\xi, J\xi)J\xi$ is in $\text{span}\{\xi, J\xi\}$. Hence, $R(\xi, J\xi, J\xi, \xi) = 0$. Now, Theorem 7 implies the assertion.

Theorem 11. Let M ($s \geq 2$, $n - s \geq 2$) be a Kaehlerian manifold of indefinite metric. If M satisfies the axiom of strongly isotropic antiholomorphic 2-planes, then M is of constant holomorphic sectional curvature.

Proof. Applying Theorem 9 we find $B(R) = 0$. Let $\{\xi, \eta\}$ be a basis of an arbitrary strongly isotropic antiholomorphic 2-plane in $T_p M$. The condition of the theorem gives $R(\xi, \eta)\eta$ is a linear combination of ξ and η . From this condition and $B(R) = 0$ we check $\rho(\xi, \xi) = 0$ for an arbitrary isotropic vector ξ . This condition implies M is Einsteinian [10]. Hence M is of constant holomorphic sectional curvature.

REFERENCES

1. Dajczer M., K. Nomizu. On sectional curvature of indefinite metrics. - Math. Ann., **247**, 1980, 279-282.
2. Schouten, J. A. Ricci Calculus. Berlin - New York, Springer, 1954.
3. Kulkarni, R. S. Curvature structures and conformal transformations. - J. Diff. Geom., **4**, 1970, 425-451.
4. Ganchev, G. Almost Hermitian manifolds similar to the complex space forms. - C. R. Acad. Bulg. Sci., **32**, 1979, 1179-1182.
5. Ganchev, G. On Bochner curvature tensor in almost Hermitian manifolds. - Pliska, Studia mathematica bulgarica, **9**, 1987, 33-43.
6. Carmen, M., A. Romero. Sur les variétés kähleriennes indéfinies à courbure sectionnelle holomorphe constante. - C. R. Acad. Paris, Sér. A, **297**, 1983, 343-344.
7. Tricerri, F., L. Vanhecke. Curvature tensors on almost Hermitian manifolds. - Trans. Amer. Math. Soc., **267**, 1981, 365-394.
8. Barros, M., A. Romero. Indefinite Kaehlerian manifolds. Math. Ann., **261**, 1982, 55-62.
9. Kassabov, O. On the axiom of planes and the axiom of spheres in the almost Hermitian geometry. - Serdica, **8**, 1982, 109-114.
10. Dajczer, M., K. Nomizu. On the boundedness of Ricci curvature of an indefinite metric. - Bol. Soc. Brasil. Mat., **11**, 1980, 25-30.

Received 10.04.1985